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Central Configurations of the Restricted Problem in E^4

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An old technique in celestial mechanics is to reduce a problem with $2n$ degrees of freedom to a problem with two degrees of freedom in order to approximate asymptotically the flow of the dynamical system. Such is the case with the reduction of the three-body problem in the plane E^2 to the (circular) restricted problem of three bodies where the motion of the primaries is independent of the two degree of freedom problem associated with the third body. One decouples the differential equations of the six degree of freedom system by choosing a known solution of the two-body problem and then by defining the two degree of freedom problem along this chosen solution. In principle we want to consider this procedure in the $(n+1)$-body problem by defining restricted problems of $n+1$ bodies that correspond to central configurations of the n-body problem. We choose to work in Euclidean space E^4 with a non-Newtonian potential to take advantage of a setting with a richer structure. However, these methods can be used in Euclidean space E^3 with the Newtonian potential and the proofs of comparable theorems follow directly. The reason we choose to study central configurations is their importance in building homographic solutions of the n-body problem.

We prove the existence of many central configurations of the restricted problem of $n+1$ bodies in Euclidean space E^4 by topological methods. We analyze the configurations by critical-point theory. Compare [2, 3].

An important reason for the study of central configurations of the restricted problem is their use in generating by induction central configurations of the $(n+1)$-body problem. The induction is carried out by using the implicit function theorem on nondegenerate central configurations of the restricted problem.

There are two examples of this use of the restricted problem in the literature. One is the induction of the Lagrange and Euler central configurations in the three-body problem from their appearance in the restricted three-body problem. The other is the induction of collinear central configurations of $n+1$ bodies from their appearance in the restricted $(n+1)$-body problem.

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We limit ourselves here to studying the existence of central configurations of a restricted $(n+1)$-body problem which is defined with respect to an arbitrarily chosen central configuration of the n-body problem. In [2] we prove comparable results for central configurations of the $(n+1)$-body problem by generalizing the methods of proof found here.

1. Restricted $n+1$-Body Problem in E^4

We define a central configuration of the restricted problem of $n+1$ bodies with respect to a central configuration of the n-body problem.

The configuration space of the n-body problem is given as a subset of $(E^4)^n$. For a given choice of the masses $(m_i) \in \mathbb{R}^n_+$, let $M - \Delta$ denote the subset of configurations which is given by

$$M - \Delta = \left\{ (x_1,..., x_n) \in (E^4)^n \;\middle|\; \sum m_i x_i = 0 \text{ and } x_i \neq x_j, i \neq j \right\}.$$

Let $S_m - \Delta \subset M - \Delta$ be the excised sphere defined by

$$S_m - \Delta = \left\{ x \in M - \Delta \;\middle|\; \tfrac{1}{2} \sum m_i \|x_i\|^2 = 1 \right\}.$$

The potential function of the n-body problem in E^4 is

$$V_m(x_1,..., x_n) = - \sum_{i<j} \frac{m_i m_j}{\|x_i - x_j\|^2}.$$

A configuration $(x_1,..., x_n) \in S_m - \Delta$ is a central configuration if and only if $(x_1,..., x_n)$ is a critical point of the potential $V_m \mid (S_m - \Delta)$ [1].

Let $(x_1,..., x_n) \in S_m - \Delta$ be a central configuration. Then it follows that the configuration satisfies

$$\lambda m_i x_i = -\text{grad}_i\, V_m(x_1,..., x_n)$$

for some $\lambda \in \mathbb{R}\backslash\{0\}$ and for all $i = 1,..., n$. Here $\text{grad}_i\, V_m$ denotes the gradient of V_m by x_i. By the homogeneity of V_m it follows that $\lambda = V_m(x_1,..., x_n)$ holds.

The potential function of the restricted problem of $n+1$ bodies with respect to the central configuration $(x_1,..., x_n) \in S_m - \Delta$ is defined by V: $E^4 - \{x_i\} \to \mathbb{R}$ where

$$V(x) = - \sum \frac{m_i}{\|x_i - x\|^2} + \lambda \frac{1}{2} \|x\|^2.$$

The constant $\lambda = V_m(x_1,\dots,x_n) < 0$ is the Lagrange multiplier. Here (and throughout the paper) we have suppressed the dependence of V on the configuration $(x_1,\dots,x_n)$.

A central configuration of the restricted problem is defined as the configuration $((x_1,\dots,x_n),x) \in S_m - \Delta \times (E^4 - \{x_i\})$, where x is a critical point of V.

We compute the linear map $DV(x)$ as

$$DV(x)(v) = -2\sum \frac{m_i}{\|x_i - x\|^4}\langle x_i - x, v\rangle + \lambda\langle x, v\rangle$$

for all $v \in E^4$. At a critical point, $x \in E^4 - \{x_i\}$, we have $DV(x) = 0$ so that x satisfies

$$\lambda x = 2\sum \frac{m_i}{\|x_i - x\|^4}(x_i - x).$$

Consequently,

$$\left(\lambda + 2\sum \frac{m_i}{\|x_i - x\|^4}\right)x = 2\sum \frac{m_i x_i}{\|x_i - x\|^4}$$

holds.

Remark. This equation has the form which x_k satisfies in the central configuration; i.e., for all $k = 1,\dots,n$, we have

$$\lambda m_k x_k = 2\sum_{i \neq k} \frac{m_i m_k}{\|x_i - x_k\|^4}(x_i - x_k).$$

PROPOSITION 1. *Let $(x_1,\dots,x_n) \in S_m - \Delta$ be a central configuration. The critical points of $V\colon E^4 - \{x_i\} \to \mathbb{R}$ lie in a compact subset of the domain.*

Proof. We must show that a critical point is bounded away from $\{\{x_i\}, \infty\}$. By the equation of the critical point

$$\lambda x = 2\sum \frac{m_i}{\|x_i - x\|^4}(x_i - x),$$

we observe that $\|x\|$ cannot be arbitrarily large. Also as $x \to x_k$ for any k, the term $\|x_k - x\|^{-3}(x_k - x)$ becomes arbitrarily large. Therefore, the equality must fail to hold for $\|x - x_k\|$ sufficiently small. Consequently, we find that $\|x\| < N$ and for each k, $\|x - x_k\| > \varepsilon$ must hold for some $N > 0$ and $\varepsilon > 0$.

PROPOSITION 2. *Let $(x_1,\dots,x_n) \in S_m - \Delta$ be a central configuration. If $V\colon E^4 - \{x_i\} \to \mathbb{R}$ is a nondegenerate function, then V has only finitely many critical points.*

This follows directly from Proposition 1 and the fact that each nondegenerate critical point is isolated.

We show the way in which the critical-point equation can be used to compute the degeneracy of a critical point x of V.

The Hessian of V at a critical point $x \in E^4 - \{x_i\}$ is the symmetric bilinear form $D^2V(x)$ which is defined as

$$D^2V(x)(v, w) = -2\sum \frac{m_i}{\|x_i - x\|^4}\left(\frac{4\langle x_i - x, v\rangle\langle x_i - x, w\rangle}{\|x_i - x\|^2} - \langle v, w\rangle\right) + \lambda\langle v, w\rangle$$

for all $(v, w) \in E^4 \times E^4$. We denote by $H_x(v, v)$ the associated quadratic form. We write $H_x(v, v)$ in the form

$$H_x(v, v) = -8\sum \frac{m_i}{\|x_i - x\|^6}\langle x_i - x, v\rangle^2 + \left(\lambda + 2\sum \frac{m_i}{\|x_i - x\|^4}\right)\|v\|^2.$$

Identify E^4 with $\mathbb{H}$, the division algebra of quaternions with basis $\{1, i, j, k\}$. For any $v \in \mathbb{H}$, let iv, jv, kv denote the vectors produced by multiplying v by i, j, k. Then $\{v, iv, jv, kv\}$ is an orthogonal set. Consequently, we have the equality

$$H_x(v, v) + H_x(iv, iv) + H_x(jv, jv) + H_x(kv, kv) = 4\lambda\|v\|^2 < 0.$$

If $x \in E^4 - \{x_i\}$ is a critical point of V, let $\operatorname{ind}(x)$ denote the index of V at x, the maximal dimension of a subspace of E^4 on which $D^2V(x)$ is negative definite. The calculation above shows that for any critical point of V, we have $\operatorname{ind}(x) \geqslant 1$. We denote by $\operatorname{rank}(x)$, the rank of V at x.

Thus we have proved the following.

THEOREM 1. *Let $x \in E^4 - \{x_i\}$ be a critical point of V. Then* $\operatorname{ind}(x) \geqslant 1$ *holds.*

COROLLARY 1.1. *If x is a critical point of V, then* $\operatorname{rank}(x) \geqslant 1$.

Our next theorems concern collinear, planar and three-dimensional central configurations in E^4. Choose a line $E^1 \subset E^4$ and let $S_1 - \Delta$ denote the intersection of $(E^1)^n$ with $S_m - \Delta$. Similarly, let $S_2 - \Delta$ and $S_3 - \Delta$ denote the intersections of $(E^2)^n$ and $(E^3)^n$ with $S_m - \Delta$. For all these theorems we fix $(m_i) \in \mathbb{R}_+^n$.

Let V_k denote the restriction $V \mid (E^k - \{x_i\})$ for $k = 1, 2, 3$. We always assume for subsequent calculations that the set $\{x_i\}$ spans E^k when we refer to the function V_k. Clearly a critical point of V_k is a critical point of V. Let $H_k(v, v)$ denote the Hessian $D^2V_k(v, v)$ for $v \in E^k$. If x is a critical point of V_k, we denote the index of x with respect to V_k by $\operatorname{ind}_k(x)$.

THEOREM 2. *Let $(x_1,..., x_n) \in S_1 - \Delta$ be a collinear central configuration so that for all i, $x_i \in E^1$. Then V has $n+1$ nondegenerate saddle points in E^1 and in any plane E^2 containing E^1, there are two maxima, nondegenerate with respect to V_2.*

THEOREM 3. *Let $(x_1,..., x_n) \in S_2 - \Delta$ be a planar (noncollinear) central configuration with $x_i \in E^2$ for all i. Then every critical point $x \in E^2 - \{x_i\}$ is a saddle point of V in $E^4 - \{x_i\}$. For any space E^3 containing E^2, there are two maxima in $E^3 \backslash E^2$, nondegenerate with respect to V_3.*

THEOREM 4. *Let $(x_1,..., x_n) \in S_3 - \Delta$ be a three-dimensional central configuration with $x_i \in E^3$ for all i. Then every critical point $x \in E^3 - \{x_i\}$ of V is a saddle point. In E^4 there are two nondegenerate maxima $x \in E^4 \backslash E^3$ of V.*

The importance of these theorems is seen by the fact that a complete description of the general critical point set is given by the nature of the central configuration as well as the imbedding space. In particular, the methods used here may be extended to the case of finite masses, the $(n+1)$-body problem [2].

Let $(x_1,..., x_n) \in S_k - \Delta$ be a central configuration and let $x \in E^k - \{x_i\}$ be a critical point of V_k. The Hessian $D^2V(x)$ splits as

$$D^2V(x) = D^2V_k(x) \oplus (D^2V_k)^{\perp}(x),$$

where $(D^2V_k)^{\perp}(x)$ is the Hessian restricted to E^{4-k} which is orthogonal to E^k.

In order to compute the degeneracies of $D^2V(x)$ we need the following facts.

Let $(x_1,..., x_n) \in S_1 - \Delta$ be a collinear central configuration so that $x_i \in E^1$ for all i. If $x \notin E^1 - \{x_i\}$, then there are two directions $v_1, v_2 \in E^4$ such that for all i, $\langle x_i - x, v_1 \rangle = 0 = \langle x_i - x, v_2 \rangle$ and $\langle v_1, v_2 \rangle = 0$.

Let $(x_1,..., x_n) \in S_2 - \Delta$ be a planar central configuration, $x_i \in E^2$ for all i. If $x \notin E^2 - \{x_i\}$, then there is a vector $v \in E^4$ such that for all i, $\langle x_i - x, v \rangle = 0$.

Let $(x_1,..., x_n) \in S_3 - \Delta$ be a three-dimensional central configuration. If $x \notin E^3 - \{x_i\}$, then for any $v \in E^4$, there is an i such that $\langle x_i - x, v \rangle \neq 0$.

2. PROOF OF THEOREM 2

Theorem 2 states that the collinear central configurations always lead to nondegenerate critical points of the potential V_2. In order to show that there are only two maxima of V_2 we proceed as follows.

PROPOSITION 3. *Let* $(x_1,..., x_n) \in S_1 - \Delta$ *be a collinear central configuration ordered on* E^1 *by* $x_1 < \cdots < x_n$. *There are always* $n + 1$ *critical points of* V_1.

Proof. Let $(x_1,..., x_n) \in S_1 - \Delta$ be a collinear central configuration. The masses $(m_i) \in \mathbb{R}_+^n$ are linearly ordered on E^1 by $x_1 < \cdots < x_n$. For any $x \in E^1 - \{x_i\}$, we see that $H(v, v) < 0$ for $v \in E^1$. Thus, every critical point of V_1 is a maximum. For m_i and m_{i+1}, two adjacent masses, $V_1 \to -\infty$ as $x \to x_i, x_{i+1}$. Thus, there is a critical point of V_1 between x_i and x_{i+1}. As every such critical point is a maximum of V_1, there is only one between every pair of adjacent masses. This accounts for $n - 1$ critical points of V_1 on E^1. But for $x < x_1$ and $x > x_n$, the same behavior occurs for $V_1 \to -\infty$ as $x \to \pm\infty$. Thus there are precisely $n + 1$ critical points of V_1 on E^1.

PROPOSITION 4. *Let* $(x_1,..., x_n) \in S_1 - \Delta$ *be a collinear central configuration; let* E^2 *be a plane that contains* E^1. *Every critical point of* V_2, $x \in E^2 \backslash E^1$, *is a nondegenerate maximum. Thus there are only two such critical points.*

Proof. Let $x \in E^2 \backslash E^1$ be a critical point of V_2. Then

$$\left(\lambda + 2 \sum \frac{m_i}{\|x_i - x\|^4}\right) x = 2 \sum \frac{m_i x_i}{\|x_i - x\|^4}$$

implies that $\lambda + 2 \sum (m_i/\|x_i - x\|^4) = 0$ must be satisfied. At this critical point we evaluate $H_2(v, v)$ as

$$H_2(v, v) = -8 \sum \frac{m_i}{\|x_i - x\|^6} \langle x_i - x, v\rangle^2 \leqslant 0$$

for every $v \in E^2$. The inequality $H_2(v, v) < 0$ holds as $\{x_i - x\}$ spans E^2. Therefore, we have proved that $x \in E^2 \backslash E^1$ is a nondegenerate maximum of V_2. Reflection across E^1 is an involution of E^2 which leaves invariant the set of critical points of V_2. As $E^2 \backslash E^1$ has two components, there are two maxima of V_2, one maximum in each component. This completes the proof.

We have shown that there are $n + 3$ critical points of V_2 in $E^2 - \{x_i\}$. There are at least two nondegenerate maxima (in $E^2 \backslash E^1$) and there are $n + 1$ critical points in $E^1 - \{x_i\}$. By a simple topological argument we show that the $n + 1$ critical points of V_2 in $E^1 - \{x_i\}$ are (possibly degenerate) saddle points with index equal to 1.

The Betti numbers of $E^2 - \{x_i\}$ are $\beta_0 = 1$ and $\beta_1 = n$. The critical points of V_2 are isolated. Thus, we may write using Morse theory that $\mu_0 - \mu_1 = \beta_0 - \beta_1 = 1 - n$, where μ_0 is the number of maxima and μ_1 is the number of saddle points. The inequality $\text{ind}(x) \geqslant 1$ shows that there is no other contribution. But from above we know that $\mu_0 + \mu_1 = 1 + n$. Thus, we obtain $\mu_0 = 2$ and $\mu_1 = n + 1$.

From these topological facts we may obtain an analytic fact as follows. Let $v \in E^2$ be orthogonal to E^1. As each $x \in E^1 - \{x_i\}$ is a saddle point, and as the Hessian splits in a pair of directions $(E^1, (E^1)^{\perp})$ it follows that $H_2(v, v) \geqslant 0$ must hold. Consequently, we have

$$\lambda + 2 \sum \frac{m_i}{\|x_i - x\|^4} \geqslant 0.$$

In order to show that each critical point $x \in E^1 - \{x_i\}$ of V_2 is a nondegenerate saddle, we must show that the Hessian is positive definite in the direction $v \in (E^1)^{\perp}$. We need the following sharp results.

PROPOSITION 5. *Let $(x_1,..., x_n) \in S_1 - \Delta$ be a collinear central configuration such that $x_i \in E^1$ for all i and $x_1 < \cdots < x_n$. The critical points of V_1, $x \in E^1$ such that $x < x_1$ and $x > x_n$ are in the intervals $(x_1 - \alpha_1, x_1)$ and $(x_n, a_n + x_n)$, where $\alpha_1^4 = -2m_1/\lambda$ and $\alpha_n^4 = -2m_n/\lambda$, $\lambda = V_m(x_1,..., x_n)$.*

Proof. Identify E^1 with $\mathbb{R}$. We write the equation for x_1 as

$$\lambda m_1 x_1 = 2 \sum_{i \neq 1} \frac{m_1 m_i}{\|x_i - x_1\|^4} (x_i - x_1).$$

Consequently, we have the inequality

$$\lambda x_1 > 2 \sum_{i \neq 1} \frac{m_i}{\|x_i - x\|^3}$$

for $x < x_1$. But $DV_1(x)$ can be written as

$$DV_1(x) = -2 \sum \frac{m_i}{\|x_i - x\|^3} + \lambda x \qquad \text{for} \quad x < x_1.$$

By setting $\alpha_1 = (-2m_1/\lambda)^{1/4}$ we find that

$$DV_1(x_1 - \alpha_1) = -2 \sum \frac{m_i}{\|x_i - x_1 + \alpha_1\|^3} + \lambda(x_1 - \alpha_1)$$

$$= -2 \sum_{i \neq 1} \frac{m_i}{\|x_i - x_1 + \alpha_1\|^3} + \lambda x_1 > 0$$

holds. But $V_1 \to -\infty$ as $x \to x_1$. Thus, the critical point $x < x_1$ of V_1 lies in the interval $(x_1 - \alpha_1, x_1)$. A similar calculation yields the result for the critical point $x > x_n$.

THEOREM 5. *Let* $(x_1,\ldots,x_n) \in S_1 - \Delta$ *be a collinear central configuration such that* $x_i \in E^1$ *for all* i *and* $x_1 < \cdots < x_n$. *Then the inequality*

$$-\lambda \leqslant 2\frac{m_i + m_{i+1}}{\|x_i - x_{i+1}\|^4}$$

is satisfied for $i = 1,\ldots, n-1$.

Proof. The equations of the critical point are

$$\lambda m_k x_k = 2 \sum_{i \neq k} \frac{m_i m_k}{\|x_i - x_k\|^4} (x_i - x_k).$$

for $k = 1,\ldots, n$. Then

$$\lambda(x_k - x_{k+1}) = 2 \sum_{i \neq k,k+1} m_i \left(\frac{x_i - x_k}{\|x_i - x_k\|^4} - \frac{x_i - x_{k+1}}{\|x_i - x_{k+1}\|^4}\right) - 2\frac{m_k + m_{k+1}}{\|x_k - x_{k+1}\|^4}(x_k - x_{k+1}).$$

Take the inner product of each side with $(x_k - x_{k+1})$. Then we have

$$-\lambda = 2\frac{m_k + m_{k+1}}{\|x_k - x_{k+1}\|^4} + \text{nonpositive terms.}$$

Consequently, $-\lambda \leqslant 2(m_k + m_{k+1})/\|x_k - x_{k+1}\|^4$ holds for all $k = 1,\ldots, n-1$.

The proof of Theorem 2 follows immediately from Theorem 5. We estimate $H_2(v, v)$, $v \in (E^1)^\perp$, $\|v\| = 1$ as

$$H_2(v, v) = \lambda + 2\sum \frac{m_i}{\|x_i - x\|^4} > 2\left(\sum \frac{m_i}{\|x_i - x\|^4} - \frac{m_k + m_{k+1}}{\|x_k - x_{k+1}\|^4}\right).$$

But for $x_k < x < x_{k+1}$ in E^1, we have

$$\frac{m_k}{\|x_k - x\|^4} + \frac{m_{k+1}}{\|x_{k+1} - x\|^4} > \frac{m_k + m_{k+1}}{\|x_k - x_{k+1}\|^4}.$$

This holds for every critical point $x \in E^1 - \{x_i\}$ of V_2, $x_1 < x < x_n$.

At the two critical points $x < x_1$ and $x > x_n$, Proposition 5 shows that $H_2(v, v) > 0$ for $v \in (E^1)^\perp$, $\|v\| = 1$. We need only use the inequalities

$$\frac{m_1}{\|x_1 - x\|^4} > -\frac{\lambda}{2}$$

for $x \in (x_1 - \alpha_1, x_1)$ and

$$\frac{m_n}{\|x_n - x\|^4} > -\frac{\lambda}{2}$$

for $x \in (x_n, a_n + x_n)$ that result from Proposition 5.

Consequently, $H_2(v, v) > 0$ for $v \in (E^1)^{\perp}$. The plane E^2 containing E^1 was selected arbitrarily. Therefore, in every direction $v \in E^4$ which is orthogonal to E^1, $H(v, v) > 0$ holds. Thus, $x \in E^1 - \{x_i\}$ is a nondegenerate saddle point of V.

This completes the proof of Theorem 2.

3. Homology Critical Points

Let us consider the case that V has only *isolated* critical points in $E^4 - \{x_i\}$. Let $c_1 < \cdots < c_r < 0$ be the (finitely many) critical values of V. Set $c_0 = -\infty$ and for any j, $1 \leqslant j \leqslant r$; define $W_j = V^{-1}(c_{j-1}, c_j)$. Let A_j be the set of critical points at the critical level c_j. For any i, $0 \leqslant i \leqslant 4$, define $\mu_i(j)$ by

$$\mu_i(j) = \operatorname{rank} H_{4-i}(W_j \cup A_j, W_j)$$

and

$$\mu_i = \sum_{j=1}^{r} \mu_i(j).$$

Then the numbers $\{\mu_i\}$ satisfy the Morse inequalities.

The same definitions may be used for the real analytic function V in case nonisolated critical points exist. There are only finitely many critical levels in this case and the strata of critical points are nondegenerate critical manifolds (by the fact that $\operatorname{ind}(x) \geqslant 1$ holds at a critical point).

In the proofs of Theorems 3 and 4 which follow we use the notation of σ_i and μ_i for the sum of homology ranks. In Theorem 3, $\{\sigma_i\}$ refers to the critical points of $V_2: E^2 - \{x_i\} \to \mathbb{R}$ and $\{\mu_i\}$ refers to the critical points of $V_3: E^3 - \{x_i\} \to \mathbb{R}$, where $(x_1, \ldots, x_n) \in S_2 - \Delta$ is a planar central configuration. In the definitions above, V and E^4 are replaced by V_k, E^k, $k = 2, 3$. In Theorem 4, $\{\sigma_i\}$ refers to the critical points of $V_3: E^3 - \{x_i\} \to \mathbb{R}$ and $\{\mu_i\}$ is defined as above, where $(x_i, \ldots, x_n) \in S_3 - \Delta$ is a three-dimensional central configuration.

4. Proof of Theorem 3

We turn now to the proof of Theorem 3. Let $(x_1,..., x_n) \in S_2 - \Delta$ be a planar central configuration with constant $\lambda = V_m(x_1,..., x_n)$. Let $x \in E^2 - \{x_i\}$ be a critical point of V_2. If x is a saddle point of V_2, then $\mathrm{ind}_2(x) = 1$ follows from $\mathrm{ind}_2(x) \geqslant 1$. Also, x is isolated (whether the saddle point is degenerate or not). If x were not isolated, then as V_2 is real analytic in $E^2 - \{x_i\}$, it follows that x must lie in a one-dimensional arc of critical points. Consequently, x would be a maximum. Therefore, the possible critical points of V_2 are (isolated) saddles, isolated maxima and circles of maxima.

Choose a space E^3 which contains E^2. Let $x \in E^3 \backslash E^2$ be a critical point of V_3. Then it follows from the critical-point equation that $\lambda + 2\sum (m_i/\|x_i - x\|^4) = 0$ holds for $x \in E^3 \backslash E^2$. The set $\{x_i - x\}$ spans E^3. Therefore, $\langle x_i - x, v\rangle \neq 0$ for every $v \in E^3$ and for some $i = 1,..., n$. We find that

$$H_3(v, v) = -8 \sum \frac{m_i}{\|x_i - x\|^6} \langle x_i - x, v\rangle^2 < 0$$

for every $v \in E^3$. Thus $x \in E^3 \backslash E^2$ is a nondegenerate maximum of V_3. As $E^3 \backslash E^2$ has two components there are two maxima, one in each component.

We take the cases (i) that the critical points of V_2 are isolated and (ii) that V_2 has nonisolated critical points.

Case (i). With the homology interpretation of Morse theory we let σ_0 be the number of maxima of V_2 and σ_1 be the number of saddle points of V_2. Each saddle point has index 1.

Let $x \in E^2 - \{x_i\}$ be a degenerate critical point of V_2. Then there is a vector $v \in E^2$ such that $H_2(v, v) = 0$. But $\{x_i - x\}$ spans E^2 for we have excluded $\{x_i\}$ collinear. Consequently, $\langle x_i - x, v\rangle \neq 0$ holds for some i. Thus $H_2(v, v) = 0$ requires that

$$\left(2 \sum \frac{m_i}{\|x_i - x^4\|} + \lambda\right) \|v\|^2 = 8 \sum \frac{m_i}{\|x_i - x\|^6} \langle x_i - x, v\rangle^2 > 0.$$

It follows that if $v \in E^4$ is orthogonal to E^2, then $H(v, v) > 0$ so that x is a degenerate saddle point of V in $E^4 - \{x_i\}$ with $\mathrm{ind}(x) = 1$.

Let $x \in E^2 - \{x_i\}$ be a nondegenerate saddle point of V_2. Then there is a $v \in E^2$ such that $H_2(v, v) > 0$. Consequently, $H(jv, jv) = H(kv, kv) > H_2(v, v) > 0$ where we have identified E^2 with $\mathbb{C} \subset \mathbb{H}$ and jv, kv represent multiplication by the units, $v = v_1 + v_2 i$.

Thus, every critical point of V_3 in $E^3 - \{x_i\}$ is isolated. Let μ_0 be the number of maxima of V_3. Let μ_1 and μ_2 be given the homology interpretation as in Section 3 with $k = 3$.

The following equations are satisfied by $\sigma_0, \sigma_1, \mu_0, \mu_1, \mu_2$.

(i) $\sigma_0 - \sigma_1 = 1 - n$,

(ii) $\mu_0 - \mu_1 + \mu_2 = 1 + n$,

(iii) $\mu_0 + \mu_1 + \mu_2 - (\sigma_0 + \sigma_1) = 2$.

Combining these equations we find that $\mu_1 = \sigma_0$ and $\mu_0 + \mu_2 = \sigma_1 + 2$.

Now σ_1 is the number of saddle points of V_2 so that each saddle point of V_2 contributes to μ_2. Thus, we find $\mu_2 \geqslant \sigma_1$. But by the existence of two maxima of V_3 in $E^3 \backslash E^2$, we have $\mu_0 \geqslant 2$. We conclude that $\mu_0 = 2$, $\mu_1 = \sigma_0$, $\mu_2 = \sigma_1$ whether or not the critical points of V_2 are degenerate. This shows again that for each critical point of V_2 we have $H(v, v) \geqslant 0$ for $v \in E^4$, orthogonal to E^2.

Case (ii). In this case we want to show that V_3 has only two maxima, both nondegenerate, even when there are (possibly) nonisolated maxima of V_3. Any degenerate critical point of V_2 in $E^2 - \{x_i\}$ is a saddle point of V_3. Using the homology interpretation of the Morse inequalities we have as in (i) that $\mu_1 = \sigma_0$ and $\mu_0 + \mu_2 = \sigma_1 + 2$. Here $\mu_0 \geqslant 2$ holds as before. Now any critical point which makes a homology contribution to σ_1 also contributes to μ_2. Thus we find that $\mu_2 \geqslant \sigma_1$. Consequently $\mu_0 = 2$ and $\mu_2 = \sigma_1$.

This completes the proof of Theorem 3.

5. Proof of Theorem 4

Finally, we prove Theorem 4. Let $(x_1, ..., x_n) \in S_3 - \Delta$ be a central configuration. As before, we conclude for each critical point $x \in E^4 \backslash E^3$ of V that $\lambda + 2 \sum (m_i / \|x_i - x\|^4) = 0$ holds. Consequently, as in the other cases, we find that every critical point $x \in E^4 \backslash E^3$ of V is a nondegenerate maximum. As there are only two components of $E^4 \backslash E^3$, we have only two such maxima.

Case (i). Isolated critical points. Let σ_0, σ_1, σ_2 be as defined in Section 3 for the contributions in homology of the critical points of V_3 in $E^3 - \{x_i\}$.

Let $\mu_0, \mu_1, \mu_2, \mu_3$ be the contributions in homology of the critical points of V in $E^4 - \{x_i\}$ as defined above.

The Betti numbers of $E^3 - \{x_i\}$ are $\beta_0 = 1$, $\beta_2 = n$ and those of $E^4 - \{x_i\}$ are $\beta_0 = 1$, $\beta_3 = n$. Consequently, $\{\sigma_i\}$ and $\{\mu_i\}$ satisfy

(i) $\sigma_0 - \sigma_1 + \sigma_2 = 1 + n$,

(ii) $\mu_0 - \mu_1 + \mu_2 - \mu_3 = 1 - n$,

(iii) $\sum \mu_i - \sum \sigma_i = 2$.

These equations lead to $\mu_0 + \mu_2 = 2 + \sigma_1$ and $\mu_1 + \mu_3 = \sigma_0 + \sigma_2$.

Every $x \in E^3 - \{x_i\}$ such that x contributes to σ_1 also contributes to μ_2. Thus $\mu_2 \geqslant \sigma_1$ holds. By the fact that $\mu_0 \geqslant 2$, we conclude that $\mu_0 = 2$ and $\mu_2 = \sigma_1$. Also every $x \in E^3 - \{x_i\}$ which contributes to σ_0 or σ_2, respectively, contributes to μ_1 or μ_3. Consequently, $\mu_1 \geqslant \sigma_0$ and $\mu_3 \geqslant \sigma_2$ hold. Thus, we conclude that $\mu_1 = \sigma_0$, $\mu_3 = \sigma_2$ and the proof is complete.

Case (ii). Nonisolated critical points. As in case (i) any critical point which makes a contribution in homology to σ_1 also contributes to μ_2. Thus we find that $\mu_2 \geqslant \sigma_1$ and $\mu_0 \geqslant 2$. Consequently, $\mu_0 = 2$ and $\mu_2 = \sigma_1$ as in the proof of Theorem 3.

This completes the proof of Theorem 4.

5. Critical Points of Regular Central Configurations

A central configuration $(x_1, \ldots, x_n) \in S_m - \Delta$ is regular in E^4 if $\{x_i\}$ spans E^4.

If $V: E^4 - \{x_i\} \to \mathbb{R}$ is a nondegenerate function, then the numbers of critical points of index k, denoted by μ_{4-k}, must be given by the Morse inequalities. The Betti numbers of the domain are $\beta_0 = 1$, $\beta_3 = n$.

We have $\sum \mu_i \geqslant 1 + n$, $\mu_0 - \mu_1 + \mu_2 - \mu_3 = 1 - n$, $\mu_4 = 0$ $(\operatorname{ind}(x) \geqslant 1)$, $\mu_0 \geqslant 1$, $\mu_1 - \mu_0 \geqslant -1$, $\mu_2 - \mu_1 + \mu_0 \geqslant 1$.

If the critical points of V are isolated, then the same inequalities hold, where the numbers μ_k are given a homology interpretation.

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